A large surgery formula for instanton Floer homology

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Joint work with Zhenkun Li

Basic idea

Knot Floer chain complex $CFK^\infty \rightsquigarrow$ Heegaard Floer homology $\widehat{HF}(S^3_m(K))$.

Instanton knot homology KHI but no differentials $\rightsquigarrow$ calculate $I^\sharp(S^3_m(K))$?

My work:

1. Construct d_+ and d_- on KHI analogous to d_w and d_z on CFK^∞;
2. Use d_+ and d_- to calculate $I^\sharp(S^3_m(K))$ for large integer m.

Conjecture (Kronheimer-Mrowka): $KHI(K) \cong \widehat{HFK}(K), I^\sharp(Y) \cong \widehat{HF}(Y)$.

Fact (Baldwin-Sivek): $\dim I^\sharp(Y) > |H_1(Y;\mathbb{Z})|$ implies the existence of irreducible $SU(2)$ representations of $\pi_1(Y)$.

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Instanton Floer homology

Suppose Y is a closed 3-manifold and $\omega \to Y$ is a Hermitian line bundle with some admissible conditions. Based on Yang-Mills equations (related to $SO(3)$ connections), Floer '88 constructed **instanton Floer homology** $I^{\omega}(Y)$.

Suppose (M, γ) is a balanced sutured manifold, where M is a 3-manifold with boundary and $\gamma \subset \partial M$ is a 1-submanifold with some balanced conditions. Kronheimer-Mrowka '10 constructed **sutured instanton homology** $SHI(M, \gamma)$.

Heegaard Floer homology

Suppose Y is a closed 3-manifold. Based on Heegaard diagrams and symplectic geometry, Ozsváth-Szabó '04 constructed **Heegaard Floer homology** $\widehat{HF}(Y), HF^{\infty}(Y), HF^{+}(Y), HF^{-}(Y)$.

Suppose $K \subset Y$ is a knot. Ozsváth-Szabó '04 and Rasmussen '03 constructed **knot Floer homology** $HFK^{\circ}(Y,K)$ for $\circ \in \{\hat{}, \infty, +, -\}$.

Suppose (M,γ) is a balanced sutured manifold. Juhász '06 constructed **sutured Floer homology** $SFH(M,\gamma)$.

Special balanced sutured manifolds

Setup	Manifold	Suture	Heegaard Floer	instanton
Sutured manifold	M	γ	SFH	SHI
Knot $K \subset Y$	$Y \backslash N(K)$	Two meridians γ_K	$\widehat{HFK}$	KHI
Closed 3-manifold Y	$Y \backslash B^3$	Connected curve δ	$\widehat{HF}$	$I^\sharp$

Instanton and Heegaard Floer homology

Conjecture (Kronheimer-Mrowka '10)

$SHI(M,\gamma) \cong SFH(M,\gamma)$.
In particular, $KHI(Y,K) \cong \widehat{HFK}(Y,K)$ and $I^\sharp(Y) \cong \widehat{HF}(Y)$.

Examples

$KHI(Y,K) \cong \widehat{HFK}(Y,K)$ holds for

- alternating links in S^3 (Kronheimer-Mrowka '11)
- all torus knots (Li-Y. '20 and Baldwin-Li-Y. '20, some partial results by Lobb-Zentner '13, Kronheimer-Mrowka '14, Hedden-Herald-Kirk '14, Daemi-Scaduto '19, *et al.*)
- all $(1,1)$-L-space knots and all constrained knots in lens spaces (Li-Y. '21).

Instanton and Heegaard Floer homology

Conjecture (Kronheimer-Mrowka '10)

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Examples

$I^\sharp(Y) \cong \widehat{HF}(Y)$ holds for

- $\Sigma_2(S^3, L)$ for any alternating link L (Scaduto '15);
- $S^3_r(K)$ for any knot K admitting lens space surgeries. (Lidman-Pinzón-Scaduto '20, Baldwin-Sivek '20);
- Seifert fibered rational homology spheres (Alfieri-Baldwin-Dai-Sivek '20);
- Strong Heegaard Floer L-spaces, i.e. $\dim \widehat{HF}(Y) = \dim \widehat{CF}(Y) = |H_1(Y;\mathbb{Z})|$ (Baldwin-Li-Y. '20).

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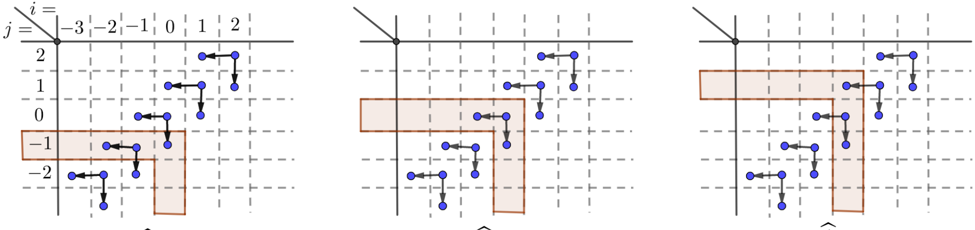
The hat version of the **bent complex** in Heegaard Floer theory:
For a knot $K \subset S^3$, choose a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$.
Let $CFK^\infty(Y,K)$ be generated by $[x,i,j] \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \times \mathbb{Z} \times \mathbb{Z}$ with the Alexander grading $A(x) = j - i$ and let the differential be

$$\partial[x,i,j] = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1\}} \#\widehat{\mathcal{M}}(\phi) \cdot [y, i - n_w(\phi), j - n_z(\phi)].$$

Let $\widehat{A}_s$ be the subcomplex generated by $[x,i,j]$ with $\max\{i, j-s\} = 0$.

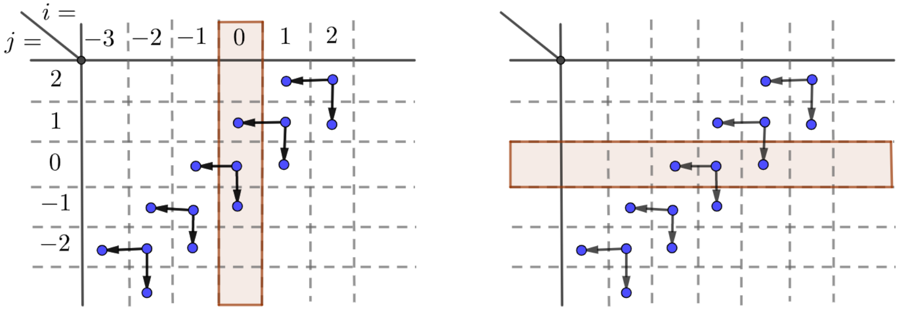


$\widehat{A}_{-1}$ $\widehat{A}_0$ $\widehat{A}_1$

Large surgery formula for Heegaard Floer homology

Since $(\widehat{CF}(S^3), d_z) = \{i=0\}, (\widehat{CF}(S^3), d_w) = \{j=0\}$, let $\hat{A}_s$ be generated by $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and let the differential d_s be

$$d_s(x) = \begin{cases} d_w(x) & A(x) > s, \\ d_w(x) + d_z(x) & A(x) = s, \\ d_z(x) & A(x) < s, \end{cases}$$



Large surgery formula for Heegaard Floer homology

Theorem (large surgery formula, Oszváth-Szabó '04, Rasmussen '03)

For integer $m >> 0$ and any integer s with $|s| \leqslant m/2$, there is an isomorphism

$$\widehat{HF}(S^3_m(K), [s]) \cong H(\hat{A}_s).$$

Here $[s] \in \mathbb{Z}/m$ is the corresponding spin^c structure on $S^3_m(K)$.

Remark

The subcomplex A_s^+ generated by $[x, i, j]$ with $\max\{i, j - s\} \geqslant 0$ computes $HF^+(S^3_m(K), [s])$.

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Main theorems

Theorem A (large surgery formula, Li-Y. '21)

There exist differentials d_+ and d_- on $KHI(-S^3,K)$ so that

$$H(KHI(-S^3,K),d_+) \cong H(KHI(-S^3,K),d_-) \cong I^\sharp(-S^3).$$

Define $A_s = (KHI(-S^3,K),d_s)$, where $d_s(x) = \begin{cases} d_+(x) & A(x) > s, \\ d_+(x) + d_-(x) & A(x) = s, \\ d_-(x) & A(x) < s, \end{cases}$

For $m >> 0$ and any s with $|s| \leqslant m/2$, there is an isomorphism

$$I^\sharp(-S^3_{-m}(K),[-s]) \cong H(A_s).$$

Here $I^\sharp(-S^3_{-m}(K)) = \bigoplus_{k=1}^{m} I^\sharp(-S^3_{-m}(K),[k])$ is a spin^c-like decomposition.

*The minus sign comes from contact gluing maps (bypass maps).

Main theorems

$\exists\, r \in \mathbb{Q}$ $\dim I^{\#}(S^3_r(K)) = |H_1(S^3_r(K))|$.

Theorem B (Li-Y. '21)

If $K \subset S^3$ is an **instanton L-space knot**, then $\dim_{\mathbb{C}} KHI(S^3, K, i) \in \{0, 1\}$, where the $\mathbb{Z}/2$-gradings of the generators of $KHI(S^3, K, i) \cong \mathbb{C}$ are alternating. Hence there exists $k \in \mathbb{N}_+$ and integers $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$ so that

$t^{n_k} - t^{n_{k-1}} + t^{n_{k-2}} - \cdots$

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j})$$

(from $\chi(KHI(K)) = \pm\Delta_K(t)$ by Lim '09, Kronheimer-Mrowka '10).

Remark

Oszváth-Szabó '05 proved an analogous result for Heegaard Floer theory. The proof of Theorem B is inspired by their proof.

Main theorems

for any r, dim $I^{\#}(S_r(K)) > |H_1(S_r(K))|$

If K is not an instanton L-space knot, then $\pi_1(S^3_r(K))$ has an irreducible $SU(2)$ representation for

1. all but finitely many slopes $r \in \mathbb{Q}\backslash\{0\}$ (Sivek-Zentner '20);
2. $r = p/q$ with p a prime power (Baldwin-Sivek '19).

Corollary A (Li-Y. '21)

The following knots are not instanton L-space knots.

1. Hyperbolic alternating knots (by Oszváth-Szabó '05);
2. Montesinos knots (including all pretzel knots), except torus knots $T(2, 2n+1)$, pretzel knots $P(-2, 3, 2n+1)$ for $n \in \mathbb{N}_+$ and their mirrors (by Baker-Moore '18).
3. Knots that are closures of 3-braids, except twisted torus knots $K(3, q; 2, p)$ with $pq > 0$ and their mirrors (by Lee-Vafaee '21).

Main theorems

Theorem C (Baldwin-Li-Sivek-Y. 21)

For any nontrivial knot $K \subset S^3$, the group of the 3-surgery $\pi_1(S^3_3(K))$ has an irreducible $SU(2)$ representation.

Remark

Kronheimer-Mrowka '04 proved the existence of representation for slope in $[0,2]$. Baldwin-Sivek '19 proved it for slope 4 and $p/q \in (2,3)$ with p a prime power. Theorem C is generalized to slope $p/q \in [16/5, 80/23) \cup (4,5)$ with p an odd prime power and $\gcd(p,5)=1$.

Theorem D (Li-Y. in preparation)

For any integer n, $I^\sharp(S^3_n(K))$ can be calculated by d_+ and d_- on $KHI(-S^3,K)$ analogous to Oszváth-Szabó's mapping cone formula for $\widehat{HF}(S^3_n(K))$.

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Analogous constructions in instanton and Heegaard Floer theory

Construction	Heegaard Floer	Instanton
Homology	$SFH(M,\gamma), \widehat{HFK}(Y,K), \widehat{HF}(Y)$	$SHI(M,\gamma), KHI(Y,K), I^\sharp(Y)$
Homological grading	Maslov grading	Relative $\mathbb{Z}/2$-grading
$\mathbb{Z}$-grading for surface S (Alexander grading)	$\langle c_1(\mathfrak{s}), [S]\rangle/2$ for spin^c structure $\mathfrak{s}$	eigenspaces of $\mu(S)$ Li '19, Ghosh-Li '19
Surgery exact triangle	Oszváth-Szabó '04	Floer '90, Scaduto '15

Analogous constructions in instanton and Heegaard Floer theory

Proposition A (surgery exact triangle, Floer '90, Scaduto '15)

Suppose K is a knot in the interior of M. Let (M_i, γ_i) be obtained from (M, γ) by Dehn surgery along K with slope μ_i. If

$$\mu_1 \cdot \mu_2 = \mu_2 \cdot \mu_3 = \mu_3 \cdot \mu_1 = -1,$$

then there exists a long exact sequence

$$SHI(M_1, \gamma_1) \to SHI(M_2, \gamma_2) \to SHI(M_3, \gamma_3) \to SHI(M_1, \gamma_1)$$

Analogous constructions in instanton and Heegaard Floer theory

Let $K \subset S^3$ be a knot and let M be the knot complement.
Suppose μ and λ are the meridian and the longidue of K.
Let $\Gamma_n \subset \partial M$ be the suture consisting of two curves of slope $-n$ (i.e. $-n\mu + \lambda$).
Push μ into $\text{int} M$ to obtain μ', with the framing induced by ∂M.

Proposition A1 (Li-Y. 20)

The $(\infty, 0, 1)$-surgery triangle on $\mu' \subset (-M, -\Gamma_n)$ induces

$$SHI(-M, -\Gamma_{n-1}) \to SHI(-M, -\Gamma_n) \to I^\sharp(-S^3) \to SHI(-M, -\Gamma_{n-1})$$

(Note that $I^\sharp(-S^3) \cong KHI(-S^3, \text{Unknot})$)
In general, let $(\hat{\mu}, \hat{\lambda}) = (\lambda - m\mu, -\mu)$ and let $\widehat{\Gamma}_n$ be the suture consisting of two curves of $-n\hat{\mu} + \hat{\lambda}$. Then $(\infty, 0, 1)$-surgery triangle on $\hat{\mu}' \subset (-M, -\widehat{\Gamma}_n)$ induces

$$SHI(-M, -\widehat{\Gamma}_{n-1}) \to SHI(-M, -\widehat{\Gamma}_n) \to I^\sharp(-S^3_{-m}(K)) \to SHI(-M, -\widehat{\Gamma}_{n-1})$$

Analogous constructions in instanton and Heegaard Floer theory

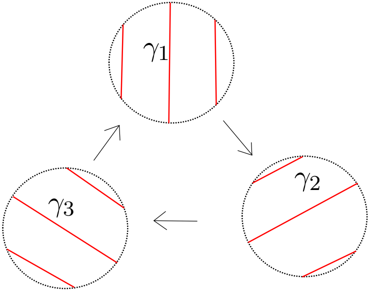
Construction	Heegaard Floer	Instanton
Homology	$SFH, \widehat{HFK}, \widehat{HF}$	$SHI, KHI, I^\sharp$
Homological grading	Maslov grading	Relative $\mathbb{Z}/2$-grading
$\mathbb{Z}$-grading for surface S (Alexander grading)	$\langle c_1(\mathfrak{s}), [S]\rangle/2$ for spin^c structure $\mathfrak{s}$	eigenspaces of $\mu(S)$ Li '19, Ghosh-Li '19
Surgery exact triangle	Oszváth-Szabó '04	Floer '90, Scaduto '15
Bypass exact triangle	Honda '00, Etnyre-Vela-Vick-Zarev '17	Baldwin-Sivek '18

Analogous constructions in instanton and Heegaard Floer theory

Proposition B (bypass exact triangle, Baldwin-Sivek '18)

Suppose $\gamma_1, \gamma_2, \gamma_3$ are three sutures on M such that γ_i are the same except in a disk, where they look like as follows. Then there exists a long exact sequence

$$SHI(-M,-\gamma_1) \to SHI(-M,-\gamma_2) \to SHI(-M,-\gamma_3) \to SHI(-M,-\gamma_1)$$



Proposition B1 (Li-Y. 20)

Let $M = S^3\backslash N(K)$ and let Γ_μ and Γ_n be the sutures of slopes μ and $-n\mu + \lambda$. Then there are two bypass exact triangles

$$\to SHI(-M, -\Gamma_{n-1}) \xrightarrow{\psi^{n-1}_{+,n}} SHI(-M, -\Gamma_n) \xrightarrow{\psi^{n}_{+,\mu}} SHI(-M, -\Gamma_\mu) \xrightarrow{\psi^{\mu}_{+,n-1}}$$

$$\to SHI(-M, -\Gamma_{n-1}) \xrightarrow{\psi^{n-1}_{-,n}} SHI(-M, -\Gamma_n) \xrightarrow{\psi^{n}_{-,\mu}} SHI(-M, -\Gamma_\mu) \xrightarrow{\psi^{\mu}_{-,n-1}}$$

Moreover, the bypass maps are homogeneous with respect to the Alexander gradings. Similarly, we can replace $\Gamma_{n-1}, \Gamma_n, \Gamma_\mu$ by $\widehat{\Gamma}_{n-1}, \widehat{\Gamma}_n, \widehat{\Gamma}_\mu$.

Analogous constructions in instanton and Heegaard Floer theory

Construction	Heegaard Floer	Instanton
Homology	$SFH, \widehat{HFK}, \widehat{HF}$	$SHI, KHI, I^\sharp$
Homological grading	Maslov grading	Relative $\mathbb{Z}/2$-grading
$\mathbb{Z}$-grading for surface S (Alexander grading)	$\langle c_1(\mathfrak{s}), [S]\rangle/2$ for spin^c structure $\mathfrak{s}$	eigenspaces of $\mu(S)$ Li '19, Ghosh-Li '19
Surgery exact triangle	Oszváth-Szabó '04	Floer '90, Scaduto '15
Bypass exact triangle	Honda '00, Etnyre-Vela-Vick-Zarev '17	Baldwin-Sivek '18
Immersed curve invariants	Hanselman-Rasmussen-Watson '16 '18	???

Analogous constructions in instanton and Heegaard Floer theory

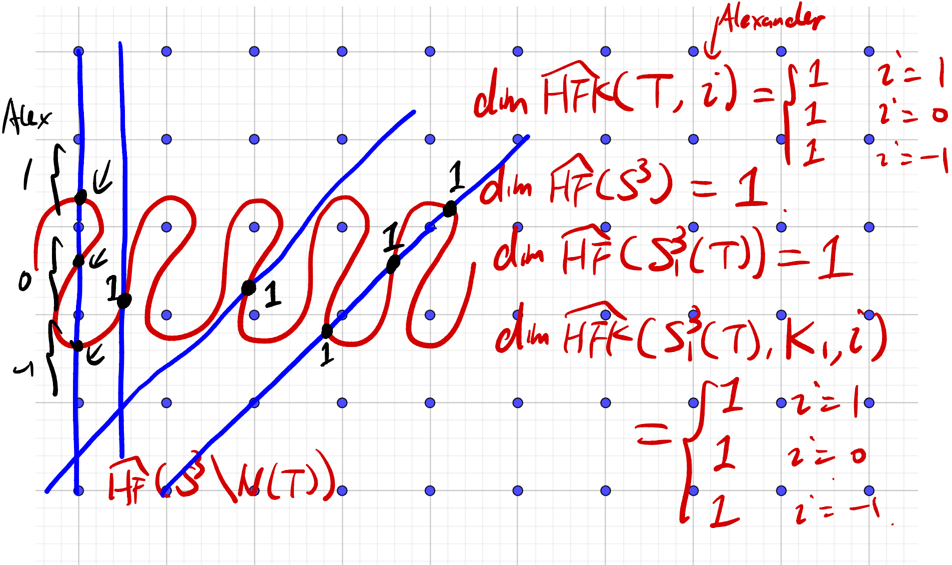
Suppose M is a 3-manifold with torus boundary. Based on Bordered Floer homology (Lipshitz-Oszváth-Thurston '08), Hanselman-Rasmussen-Watson '16 constructed a set of immersed curves in $\partial M \backslash \mathrm{pt}$.

It is denoted by $\widehat{HF}(M)$ and can be regarded as an object in some Fukaya category of $\partial M \backslash \mathrm{pt}$. If $Y = M_1 \cup_{T^2} M_2$, then

$$\dim \widehat{HF}(Y) = \dim HF_{symp}(\widehat{HF}(M_1), \widehat{HF}(M_2)) = |\widehat{HF}(M_1) \cap \widehat{HF}(M_2)|.$$

In particular, when $M = S^3 \backslash N(K)$, we can recover $\widehat{HF}(S^3_r(K))$ and $\widehat{HFK}(S^3_r(K), K_r)$ as follows, where K_r is the dual knot.

Immersed curves in the universal cover of $\partial M \cong T^2$



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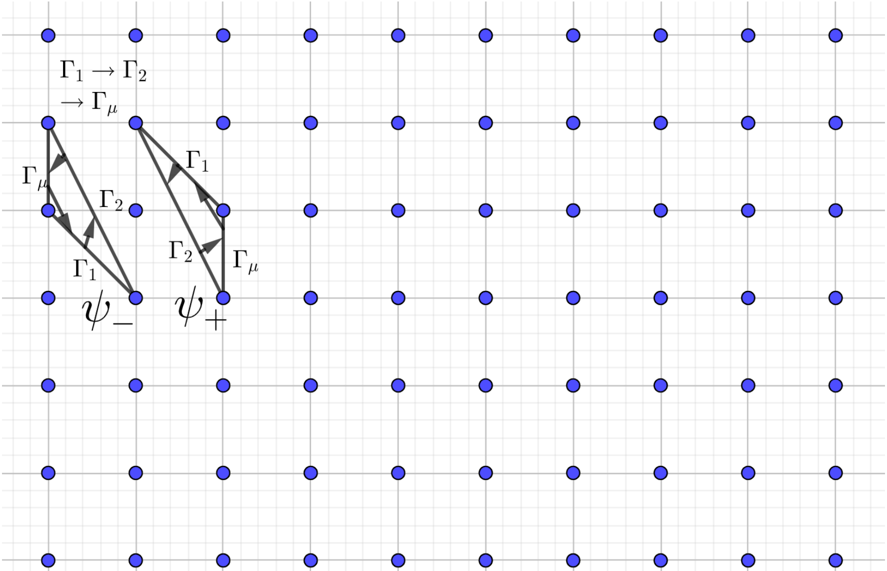

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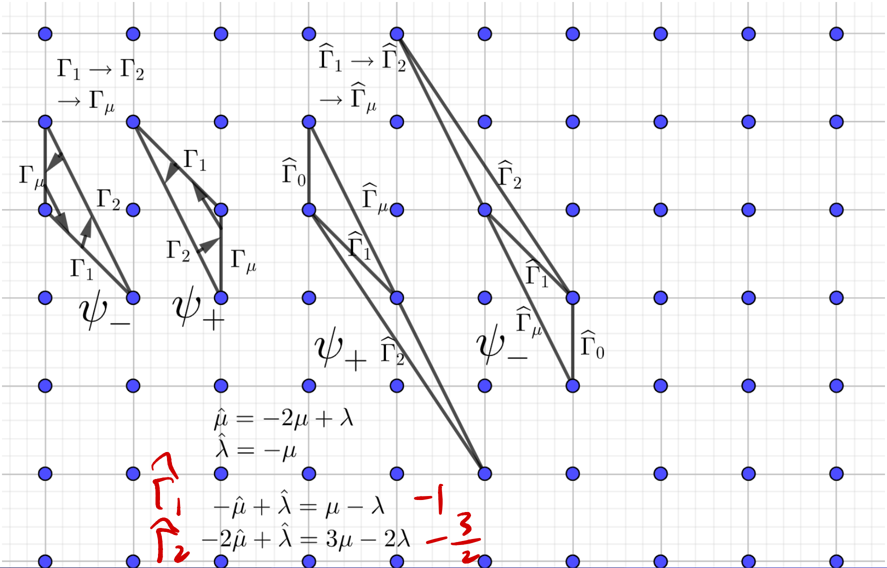

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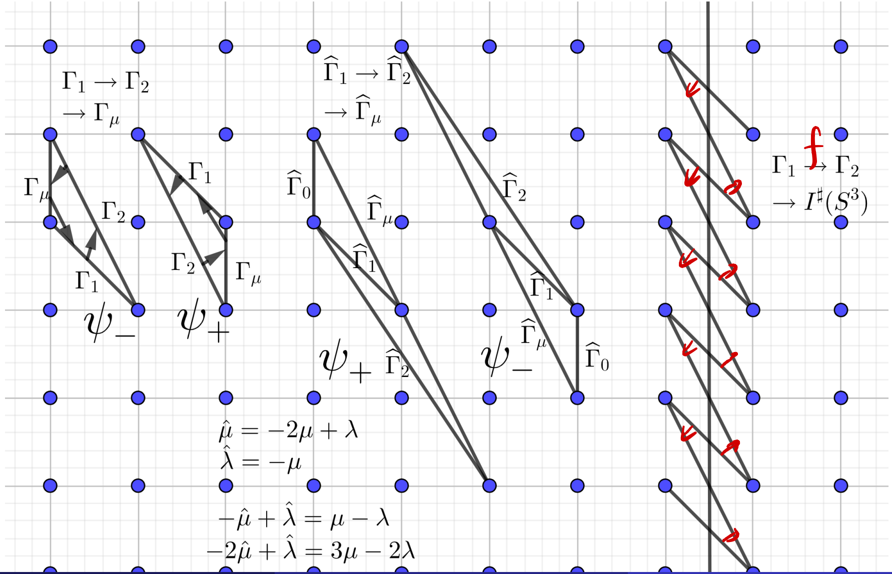
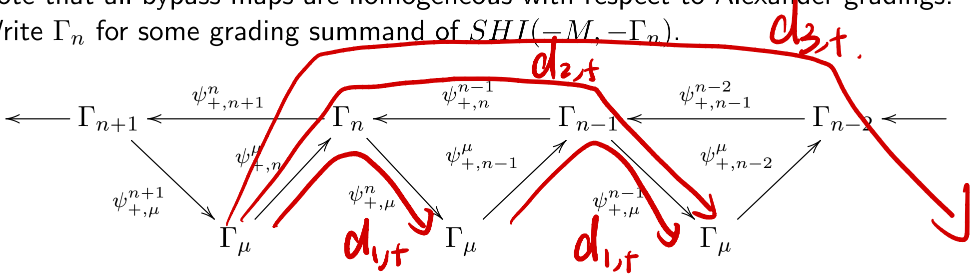


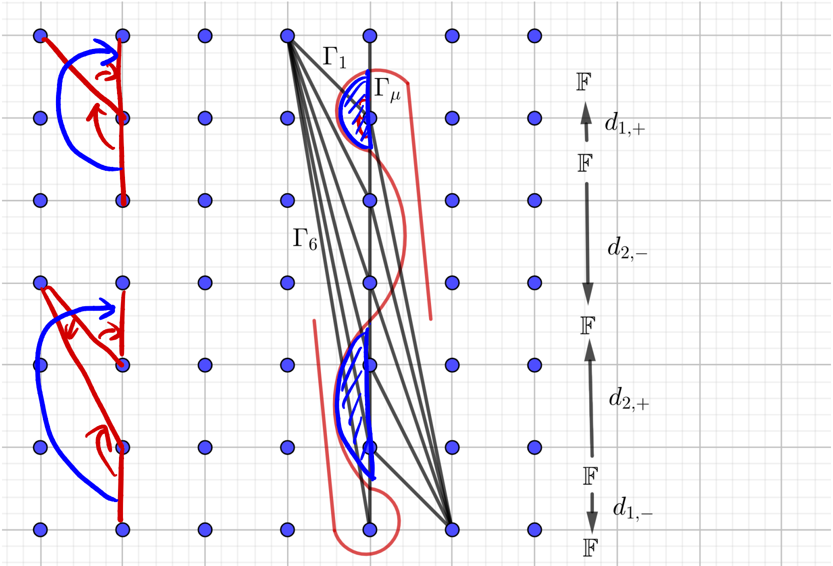
Note that all bypass maps are homogeneous with respect to Alexander gradings. Write Γ_n for some grading summand of $SHI(-M,-\Gamma_n)$.



Define $d_{1,+}=\psi^n_{+,\mu}\circ\psi^\mu_{+,n}$; $\quad d_{2,+}=\psi^{n-1}_{+,\mu}\circ(\psi^{n-1}_{+,n})^{-1}\circ\psi^\mu_{+,n}$;
$d_{r,+}=\psi^{n-r+1}_{+,\mu}\circ(\psi^{n-r+2}_{+,n-r+1})^{-1}\circ\cdots\circ(\psi^{n-1}_{+,n})^{-1}\circ\psi^\mu_{+,n}$. Then We have

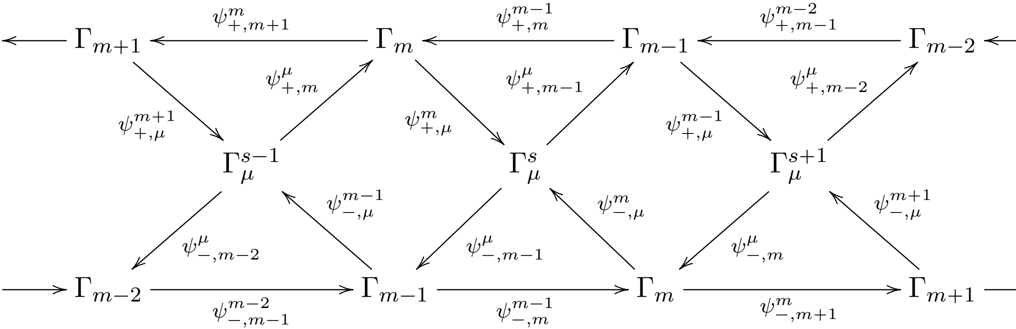
1. $d_{r,+}$ is independent of n
2. $d_{r_1,+}\circ d_{r_2,+}=0$ for any $r_1,r_2\geqslant 1$, hence $d_+^2=(\sum_r d_{r,+})^2=0$
3. $d_{r,+}$ increases the Alexander grading by r
4. $H(SHI(-M,-\Gamma_\mu),d_+)\cong I^\sharp(-S^3)$

Immersed curves in the universal cover of $\partial M \cong T^2$



Analogous constructions in instanton and Heegaard Floer theory

Indeed, we have two spectral sequences associated to $d_{r,+}$ and $d_{r,-}$.
Set $n = m$. Then we can construct A_s as follows.



Sketch of the proof of the large surgery formula

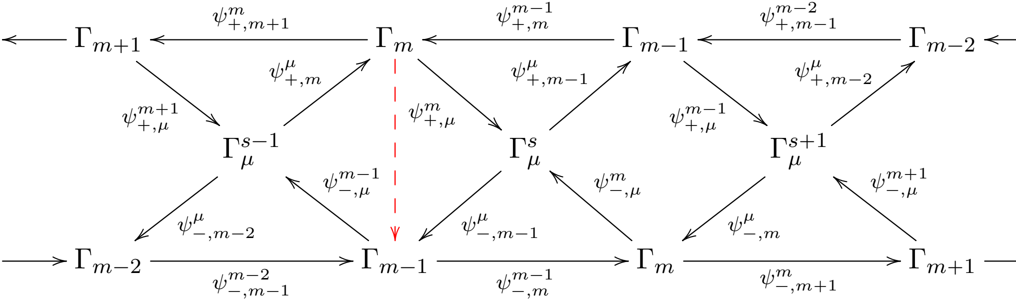
Step 1. Suppose $m >> 0$ and $\hat{\mu} = -m\mu + \lambda$. Then the slope of $\widehat{\Gamma}_2$

$$-2\hat{\mu} + \hat{\lambda} = -2(-m\mu + \lambda) + (-\mu) = (2m-1)\mu - 2\lambda \quad \frac{2m-1}{-2}$$

is large enough so that we can use 'middle Alexander gradings' of $SHI(-M, -\widehat{\Gamma}_2)$ to recover the information of $I^{\sharp}(-S^3_{-m}(K), [s])$.

Step 2. The bypass exact triangle induces a long exact sequence

$$\to SHI(-M,-\Gamma_m) \xrightarrow{\psi^{\mu}_{-,m-1}\circ\psi^{m}_{+,\mu}} SHI(-M,-\Gamma_{m-1}) \to SHI(-M,-\widehat{\Gamma}_2) \to$$



Sketch of the proof of the large surgery formula

Step 1. Suppose $m >> 0$ and $\hat{\mu} = -m\mu + \lambda$. Then the slope of $\widehat{\Gamma}_2$

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is large enough so that we can use 'middle Alexander gradings' of $SHI(-M, -\widehat{\Gamma}_2)$ to recover the information of $I^\sharp(-S^3_{-m}(K), [s])$.

Step 2. The bypass exact triangle induces a long exact sequence

$$\to SHI(-M, -\Gamma_m) \xrightarrow{\psi^{\mu}_{-,m-1} \circ \psi^{m}_{+,\mu}} SHI(-M, -\Gamma_{m-1}) \to SHI(-M, -\widehat{\Gamma}_2) \to$$

Step 3. Use the octahedral axiom (TR 4) to prove isomorphisms $H(A_s) \xrightarrow{\text{TR4}} H(\text{Cone}(\psi^{\mu}_{-,m-1} \circ \psi^{m}_{+,\mu})) \xrightarrow{\text{Step2}} SHI(-M, -\widehat{\Gamma}_2, s') \xrightarrow{\text{Step1}} I^\sharp(-S^3_{-m}(K), [-s])$.

Analogous constructions in instanton and Heegaard Floer theory

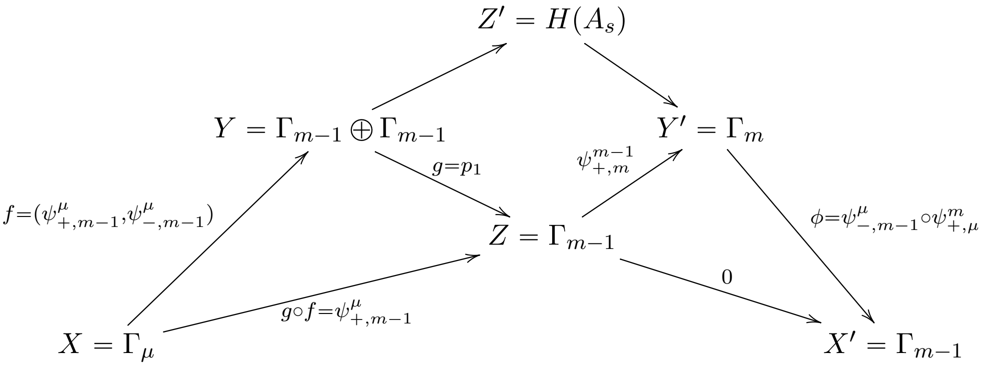
Further directions:

Construction	Heegaard Floer	Instanton
Homology	$SFH, \widehat{HFK}, \widehat{HF}$	$SHI, KHI, I^\sharp$
Large surgery formula	Oszváth-Szabó '04	Li-Y. '21
Mapping cone formula	Oszváth-Szabó '08 '11	Li-Y. in preparation
Bordered Floer homology	Lipshitz-Oszváth-Thurston '08	???
Immersed curve invariants	Hanselman-Rasmussen-Watson '16 '18	???

Thanks for your attention.

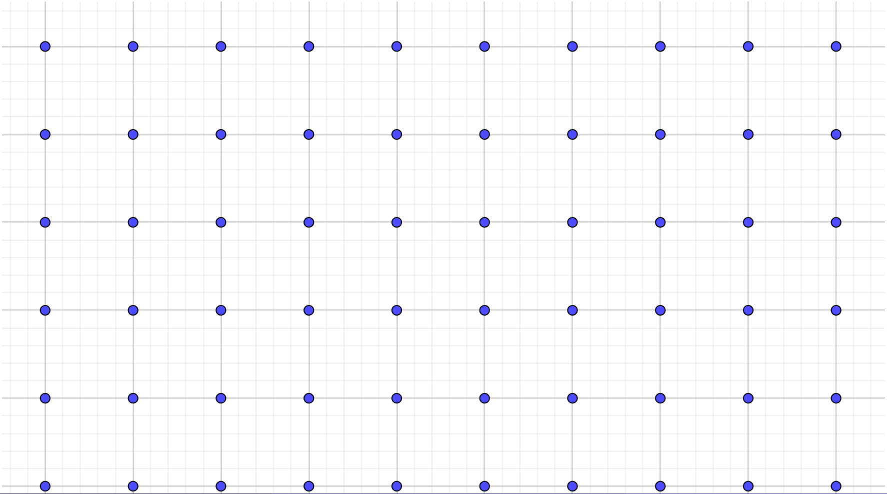
The octahedral axiom

Suppose X, Y, Z, X', Y', Z' are graded spaces. Then three long exact sequences about $f, g, g \circ f$ induce the fourth one about Z', Y', X'.



Immersed curves in the universal cover of $\partial M \cong T^2$

Note: Fukaya category is also a triangulated category so also satisfies the octahedral axiom.



Analogous constructions in instanton and Heegaard Floer theory

Construction	Heegaard Floer	Instanton
Homology	$SFH, \widehat{HFK}, \widehat{HF}$	$SHI, KHI, I^\sharp$
Minus version	Reconstruction of HFK^- Etnyre-Vela-Vick-Zarev '17	$\underline{\mathrm{KHI}}^-$ Li '19

Analogous constructions in instanton and Heegaard Floer theory

Theorem (Etnyre-Vela-Vick-Zarev '17)

The direct limit of the following system is isomorphic to $HFK^-(-S^3, K)$

$$SFH(-M, -\Gamma_{n-1}) \xrightarrow{\psi^{n-1}_{-,n}} SFH(-M, -\Gamma_n) \xrightarrow{\psi^{n}_{-,n+1}} SFH(-M, -\Gamma_{n+1}) \xrightarrow{\psi^{n+1}_{-,n+2}}$$

The maps $\{\psi^n_{+,n-1}\}$ induce the U-action on $HFK^-(-S^3, K)$.

Definition (Li '19)

Let $\underline{\mathrm{KHI}}^-(-S^3, K)$ be the direct limit of

$$SHI(-M, -\Gamma_{n-1}) \xrightarrow{\psi^{n-1}_{-,n}} SHI(-M, -\Gamma_n) \xrightarrow{\psi^{n}_{-,n+1}} SHI(-M, -\Gamma_{n+1}) \xrightarrow{\psi^{n+1}_{-,n+2}}$$

Then the maps $\{\psi^n_{+,n-1}\}$ induce the U-action on $\underline{\mathrm{KHI}}^-(-S^3, K)$.

Moreover, we can replace $\Gamma_{n-1}, \Gamma_n, \Gamma_\mu$ by $\widehat{\Gamma}_{n-1}, \widehat{\Gamma}_n, \widehat{\Gamma}_\mu$ to define $\underline{\mathrm{KHI}}^-(-S^3_{-m}(K), K_{-m})$ for the dual knot K_{-m}.

Analogous constructions in instanton and Heegaard Floer theory

Note that for $s << 0$, we have $HFK^-(-S^3, K, s) \cong \widehat{HF}(-S^3)$ and $HFK^-(-S^3_{-m}(K), K_{-m}, s) \cong \widehat{HF}(-S^3_{-m}(K), [s - s_0])$ for some s_0.

Proposition (Li-Y. '20)

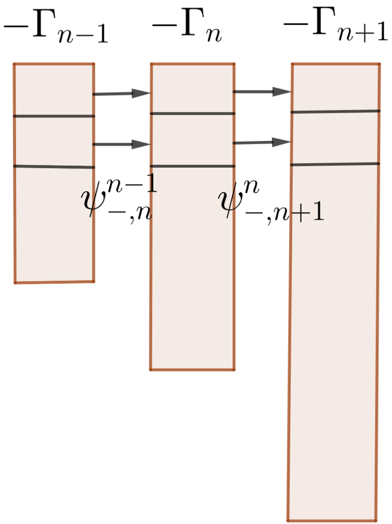
For $s << 0$, we have

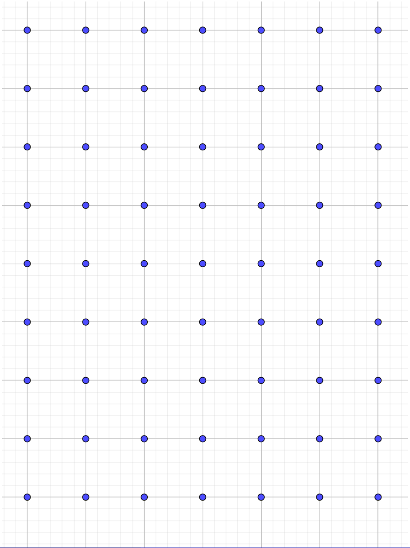
$$\bigoplus_{k=1}^{m} \underline{\mathrm{KHI}}^-(-S^3_{-m}(K), K_{-m}, s+k) \cong I^\sharp(-S^3_{-m}(K)).$$

Hence we can define $I^\sharp(-S^3_{-m}(K), [s+k])$ by $\underline{\mathrm{KHI}}^-(-S^3_{-m}(K), K_{-m}, s+k)$.

Since the direct system to define $\underline{\mathrm{KHI}}^-$ stabilizes for any fixed Alexander grading, we can also use 'middle gradings' of $SHI(-M, -\widehat{\Gamma}_n)$ for any $n >> 0$ to define the spin^c-like decomposition of $I^\sharp(-S^3_{-m}(K))$.

Diagram of the direct system





Analogous constructions in instanton and Heegaard Floer theory

Construction	Heegaard Floer	Instanton
Homology	$SFH, \widehat{HFK}, \widehat{HF}$	$SHI, KHI, I^{\sharp}$
Minus version	Reconstruction of HFK^- Etnyre-Vela-Vick-Zarev '17	$\underline{\mathrm{KHI}}^-$ Li '19
Decomposition	(torsion) spin^c structures	along $H_1(M;\mathbb{Z})$, Li-Y. '21
Euler characteristic	$\chi(SFH(M,\gamma)) = \tau(M,\gamma)$, Friedl-Juhász-Rasmussen '09, partial results by Oszváth-Szabó '04 '08	$\chi(SHI(M,\gamma)) = \tau(M,\gamma)$, Li-Y. 21, partial results by Lim '09, Kronheimer-Mrowka '10, Scaduto '15

Analogous constructions in instanton and Heegaard Floer theory

Theorem (Li-Y. 21)

For a balanced sutured manifold (M, γ) with $H = H_1(M;\mathbb{Z})$, we have a (possibly noncanoical) decomposition $SHI(M,\gamma) = \bigoplus_{h\in H} SHI(M,\gamma,h)$. Define the Euler characteristic

$$\chi(SHI(M,\gamma)) = \sum_{h\in H} \chi(SHI(M,\gamma,h)) \cdot h \in \mathbb{Z}[H]/\pm H.$$

Then we have $\chi(SHI(M,\gamma)) = \chi(SFH(M,\gamma)) = \tau(M,\gamma) \in \mathbb{Z}[H]/\pm H$.

Remark

The decomposition associated to the nontorsion part of H comes from the Alexander grading, and the torsion part comes from the 'middle gradings' of Γ_n for $n >> 0$.